# ON THE PURELY ELASTIC IMPACT OF MATERIAL SYSTEMS

(OB ABSOLIUTNO UPRUGOM UDARE MATERIAL'NYKH SISTEM)

*PMM Vol.24, No.5, 1960, pp. 781-789*

V.I. KIRGETOV
(Moscow)

*(Received 8 June 1960)*

An impact resulting from the application of a one-sided constraint upon a system is considered from a general point of view in this paper.

The analytical theory of the impact presented is based on the principle of D'Alembert-Lagrange. It stems from the fact that the applied one-sided constraint generates auxiliary limitations on "admissible displacements" of a system, and is built on the assumption that the basic mechanics equation remains valid and that the applied constraint is permanently attached to the system during impact.

**1.** A material system of $n$ points with masses $m_i$ and coordinates $x_i$, $y_i$, $z_i$ relative to some stationary Cartesian system of coordinates is given. The points are constrained by smooth holonomic time-independent constraints, the equations of which are

$$f_\alpha(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = 0 \tag{1.1}$$

"Admissible displacements" for the system are determined by the relationships

$$\sum \left(\frac{\partial f_\alpha}{\partial x_i}\delta x_i + \frac{\partial f_\alpha}{\partial y_i}\delta y_i + \frac{\partial f_\alpha}{\partial z_i}\delta z_i\right) = 0 \tag{1.2}$$

A smooth time-independent one-sided constraint

$$\varphi(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) \geqslant 0 \tag{1.3}$$

is imposed upon the system at some instant during motion.

Then an impact occurs in the system along with certain narrowing in the diversity of the "admissible displacements" by virtue of the auxiliary limitation generated by the constraint (1.3):

$$\sum\left(\frac{\partial\varphi}{\partial x_i}\,\delta x_i+\frac{\partial\varphi}{\partial y_i}\,\delta y_i+\frac{\partial\varphi}{\partial z_i}\ z_i\right)\geqslant 0 \tag{1.4}$$

Let us assume that for the duration of the impact the constraint (1.3) remains permanently attached to the system. This means that during the impact condition (1.4) must be satisfied. Also assume that the basic mechanics equation is fully valid throughout the impact.

We will consider the idealized scheme of the impact. In order to do this we will integrate the basic mechanics equation within the region of impact duration and let the duration tend to zero. During the impact there are considerable changes in system velocities for comparatively insignificant changes in position and, therefore, in its "admissible displacements". Taking this into account we obtain in the limit

$$\sum m_i\left(\Delta x_i'\delta x_i+\Delta y_i'\delta y_i+\Delta z_i'\,\delta z_i\right)\geqslant 0 \tag{1.5}$$

*Note*. Here and elsewhere primes denote differentiation with respect to time and $\Delta$ denotes differences in value for the corresponding quantities after and immediately before the impact. It is worth noting also that the constants and functions remaining unchanged during the impact are freely moved inside and outside $\Delta$. This property of $\Delta$ is widely used below.

It can easily be seen that the conditions (1.4) and (1.5), together with the equations for system constraints, are insufficient for finding the state of the system after an impact with its state prior to impact known. One condition is missing for a single-valued solution of the proposed problem. The equation for a one-sided constraint (1.3) is not useful for this purpose precisely in view of its one-sidedness. The missing condition must therefore be taken "from outside".

Let us take as such a condition the conservation of kinetic energy in the system

$$\Delta T=0 \tag{1.6}$$

Conditions (1.4) and (1.5), together with (1.6), constitute a closed system of conditions for a purely elastic impact.

**2.** Whatever the impact (elastic, inelastic, with friction or without) the state of the system must be kinematically admissible after impact. For this reason no system of impact conditions may be considered acceptable if the final state of the system defined by these conditions only is incompatible with the constraints (including, of course, one-sided constraints). Let us verify this important condition in our case. However, only the inequality (1.4) is to be investigated.

*Note.* For convenience of presentation the continuous numbering of variables will be used in this section ($x_1$, $x_2$, $x_3$, $m_1 = m_2 = m_3$ are coordinates and the mass of the first point in system; $x_4$, $x_5$, $x_6$, $m_4 = m_5 = m_6$ are coordinates and the mass of the second point, etc.).

Let us start with the derivation of explicit expressions for the values of velocity variations in system points. To this end we take the equality sign in (1.4). Then it is easy to see that in (1.5) we should also take the equality sign (it will be shown below that if (1.4) is considered an inequality then (1.5) becomes valid automatically). We obtain

$$\sum m_i \Delta x_i' \delta x_i = 0, \qquad \sum \frac{\partial \varphi}{\partial x_i} \delta x_i = 0, \qquad \sum \frac{\partial f_\alpha}{\partial x_i} \delta x_i = 0 \tag{2.1}$$

Multiplying the second and the consecutive equalities by the indetermined multipliers $\mu$ and $\lambda_\alpha$, adding the first equality and using classical reasoning, we obtain

$$m_i \Delta x_i' + \sum \frac{\partial f_\alpha}{\partial x_i} \lambda_\alpha + \frac{\partial \varphi}{\partial x_i} \mu = 0 \tag{2.2}$$

Using Equations (1.1) for bilateral constraints of the system, we express the multipliers $\lambda_\alpha$ in terms of $\mu$. From (1.1) we find

$$\sum \frac{\partial f_\alpha}{\partial x_i} \Delta x_i' = 0$$

Substituting for $\Delta x_i'$ their expressions from (2.2) and applying the notation

$$a_{\alpha\beta} = \sum \frac{1}{m_i} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_i}, \qquad a_\alpha = \sum \frac{1}{m_i} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial \varphi}{\partial x_i}$$

we obtain

$$\sum a_{\alpha\beta} \lambda_\beta + \mu a_\alpha = 0, \quad \text{или} \quad \lambda_\beta = -\mu \sum \frac{A_{\beta\alpha}}{A} a_\alpha \tag{2.3}$$

Here $A_{\beta\alpha}$ denotes the algebraic supplement to the element $a_{\alpha\beta}$ in the determinant $|a_{\alpha\beta}| = A$. We will prove that the determinant is non-zero and that consequently the last transformation is permissible.

Indeed, let

$$\sum a_{\alpha\beta} c_\alpha = 0$$

where not all $c_\alpha$ are zero. Then

$$\sum c_\alpha a_{\alpha\beta} = \sum c_\alpha \frac{1}{m_i} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_i} = \sum \frac{\partial f_\beta}{\partial x_i} \sum \frac{c_\alpha}{m_i} \frac{\partial f_\alpha}{\partial x_i} = 0$$

These equalities can be expressed in the following form:

$$\sum \frac{\partial f_\beta}{\partial x_i} u_i = 0, \qquad u_i = \sum \frac{c_\alpha}{m_i} \frac{\partial f_\alpha}{\partial x_i}$$

Hence

$$\sum m_i u_i^2 = \sum c_\alpha \sum \frac{\partial f_\alpha}{\partial x_i} u_i = 0$$

i.e. $u_i = 0$. Thus

$$\sum c_\alpha \frac{\partial f_\alpha}{\partial x_i} = 0$$

where not all $c_\alpha$ are zero. But this is impossible by virtue of the system constraint equations (1.1) which have been assumed independent. This is as was required.

The multipliers $\lambda_\alpha$ can now be excluded from the equalities (2.2). Substituting the expressions from (2.3) for $\lambda_\alpha$ into (2.2) we obtain

$$\Delta x_i' = \mu R_i, \qquad R_i = \frac{1}{m_i} \left( \sum \frac{\partial f_\alpha}{\partial x_i} A_{\alpha\beta}\, a_\beta \frac{1}{A} - \frac{\partial \varphi}{\partial x_i} \right) \tag{2.4}$$

The equalities (2.4) provide the explicit expressions for $x_i'$ in terms of the indetermined multiplier $\mu$. In order to find $\mu$ we will use condition (1.6) for the conservation of kinetic energy during impact. If we denote by $x_{i0}'$ and $x_i'$, respectively, the $i$th component of system velocity immediately before and after impact then condition (1.6) can be written as

$$\sum m_i (x_i' + x_{i0}')\, \Delta x_i' = 0$$

Eliminating from this the quantities $\Delta x_i'$ with the use of equalities (2.4) and noting that during impact $\mu \neq 0$ we obtain

$$\sum m_i (x_i' + x_{i0}')\, R_i = 0 \tag{2.5}$$

The last equality may in turn be written as

$$\sum m_i (\Delta x_i' + 2x_{i0}')\, R_i = 0$$

Substituting here the expressions for $\Delta x_i'$ from (2.4) we obtain

$$\mu \sum m_i R_i^2 + 2 \sum m_i R_i x_{i0}' = 0$$

And since

$$\sum \frac{\partial f_\alpha}{\partial x_i} x_{i0}' = 0 \tag{2.6}$$

we have

$$\sum m_i R_i x_{i0}' = \sum x_{i0}' \left( \sum \frac{\partial f_\alpha}{\partial x_i} \frac{A_{\alpha\beta}}{A} a_\beta - \frac{\partial \varphi}{\partial x_i} \right) = - \sum \frac{\partial \varphi}{\partial x_i} x_{i0}'$$

Consequently

$$\mu \sum m_i R_i^2 = 2 \sum \frac{\partial \varphi}{\partial x_i} x_{i0}', \qquad \text{or} \qquad \mu = 2 \sum \frac{\partial \varphi}{\partial x_i} x_{i0}' \Big/ \sum m_i R_i^2$$

Now it is easy to establish that the state of the system after impact is kinematically admissible.

Indeed, if in the equality (2.5) the expression for $R_i$ is expanded and, using equalities (2.6) as well as the analogous equalities for the system velocities after impact, we obtain

$$\sum \frac{\partial \varphi}{\partial x_i} x_i' = - \sum \frac{\partial \varphi}{\partial x_i} x_{i0}'$$

Hence if it is taken into account that prior to impact

$$\sum \frac{\partial \varphi}{\partial x_i} x_{i0}' < 0$$

(it is because of this circumstance that the impact occurs), it follows that

$$\sum \frac{\partial \varphi}{\partial x_i} x_i' > 0$$

which is as required.

In conclusion we will show that if the condition (1.4) is taken as an inequality, then condition (1.5) becomes valid automatically. Indeed

$$\sum m_i \Delta x_i' \delta x_i = \sum m_i \mu R_i \delta x_i = \mu \sum \delta x_i \left( \sum \frac{\partial f_\alpha}{\partial x_i} \frac{A_{\alpha\beta}}{A} a_\beta - \frac{\partial \varphi}{\partial x_i} \right) =$$

$$= - \mu \sum \frac{\partial \varphi}{\partial x_i} \delta x_i = - 2 \sum \frac{\partial \varphi}{\partial x_i} x_{i0}' \sum \frac{\partial \varphi}{\partial x_i} \delta x_i \Big/ \sum m_i R_i^2 \geqslant 0 \quad .$$

since the product in the numerator of the fraction is always negative. Thus the existence of inequalities in conditions (1.4) and (1.5) is not of essential importance for the theory presented.

**3.** Let us consider now certain general properties of the impact occurring during the application of a single constraint upon a system.

It is possible that among the "admissible displacements" of the system at the moment of impact there is a two-sided translational displacement of the system as a rigid body along some direction $\lambda$. In such a case, substituting in condition (1.5) the values

$$\delta x_i = \alpha l, \qquad \delta y_i = \beta l, \qquad \delta z_i = \gamma l$$

where $\alpha$, $\beta$, $\gamma$ are the direction cosines of $\lambda$ relative to a coordinate system, and $l$ is an arbitrary positive or negative number, we will obtain (in view of the arbitrary sign of $l$)

$$\sum m_i (\alpha \Delta x_i' + \beta \Delta y_i' + \gamma \Delta z_i') = 0$$

From this we easily derive

$$\Delta (\alpha v_x + \beta v_y + \gamma v_z) = 0$$

where $v_x$, $v_y$, $v_z$ are the components of the system's center-of-gravity velocities.

Thus, if at the moment of impact the system constraints admit a two-sided translational motion of the system as a rigid body in any direction, then the impact of the system is not reflected on the velocity of its center of gravity in this direction.

Let us assume now that at the instant of impact there is a two-sided rotation of the system as a rigid body about some axis among its "admissible displacements".

For this case we can place in the conditions (1.5)

$$\delta x_i = (\beta \zeta_i - \gamma \eta_i)\, \delta\varphi, \quad \delta y_i = (\gamma \xi_i - \alpha \zeta_i)\, \delta\varphi, \quad \delta z_i = (\alpha \eta_i - \beta \xi_i)\, \delta\varphi$$

where $\delta\phi$ is an elementary positive or negative rotation of the system about the axis $\lambda$; $\alpha$, $\beta$, $\gamma$ are the angular coefficients of this axis; $\xi_i$, $\eta_i$, $\zeta_i$ are the coordinates of the $i$th point of system in the system of coordinates with the origin at $\lambda$ and the axes parallel to the axes of the basic Cartesian system of coordinates.

The equality derived from (1.5)

$$\sum m_i \left[\Delta x_i' (\beta\zeta_i - \gamma\eta_i) + \Delta y_i' (\gamma\xi_i - \alpha\zeta_i) + \Delta z_i' (\alpha\eta_i - \beta\xi_i)\right] = 0$$

can be easily transformed into the form

$$\Delta (\alpha K_\xi + \beta K_\eta + \gamma K_\zeta) = 0 \tag{3.1}$$

where $K_\xi$, $K_\eta$, $K_\zeta$ are the system's moments of momenta about the axes $\xi$, $\eta$, $\zeta$.

The equality (3.1) means that the moment of momentum for the system relative to $\lambda$ is invariant.

Thus, if at the instant of impact the system constraints permit an elementary two-sided rotation of the system as a rigid body about some axis, then the impact of the system is not reflected on the magnitude of the system's moment of momentum relative to this axis.

The above-established theorems are in reference to a complete system. However, they remain valid for any part of the system for which the requirements of these theorems are satisfied with the assumption that the components of "admissible displacements" in the remaining parts of the system are all set equal to zero.

For example, let there be given two material points one of which strikes a fixed plane. The striking point may be translated along the surface for zero "admissible displacement" of the other point. Therefore, the particular theorem on the center-of-gravity motion is applicable to it, from which it is immediately seen that the tangent comprising the velocity of the striking point is not changing during the impact.

**4.** We shall write the conditions for purely elastic impact in the Lagrangian coordinates.

Let $q_1, \ldots, q_m$ be Lagrangian coordinates of a system. Then

$$x_i = x_i(q_1, \ldots, q_m), \qquad y_i = y_i(q_1, \ldots, q_m), \qquad z_i = z_i(q_1, \ldots, q_m) \tag{4.1}$$

$$\delta x_i = \sum \frac{\partial x_i}{\partial q_\alpha} \delta q_\alpha, \qquad \delta y_i = \sum \frac{\partial y_i}{\partial q_\alpha} \delta q_\alpha, \qquad \delta z_i = \sum \frac{\partial z_i}{\partial q_\alpha} \delta q_\alpha \tag{4.2}$$

With the aid of Expressions (4.2) we eliminate from the left part of condition (1.5) the quantities $\delta x_i$, $\delta y_i$, $\delta t_i$ and rearrange it:

$$\sum m_i (\Delta x_i' \delta x_i + \Delta y_i' \delta y_i + \Delta z_i' \delta z_i) = \sum m_i \left(\Delta x_i' \sum \frac{\partial x_i}{\partial q_\alpha} \delta q_\alpha + \cdots\right)$$

$$= \sum \delta q_\alpha \sum m_i \Big(\Delta x_i' \frac{\partial x_i}{\partial q_\alpha} + \cdots\Big) = \sum \delta q_\alpha \Delta \sum m_i \Big(x_i' \frac{\partial x_i}{\partial q_\alpha} + \cdots\Big)$$

$$= \sum \delta q_\alpha \Delta \sum m_i \Big(x_i' \frac{\partial x_i'}{\partial q_\alpha'} + \cdots\Big) = \sum \Delta \frac{\partial T}{\partial q_\alpha'} \delta q_\alpha$$

Thus, in the Lagrangian coordinates condition (1.5) is

$$\sum \Delta \left(\frac{\partial T}{\partial q_\alpha'}\right) \delta q_\alpha \geqslant 0 \tag{4.3}$$

Condition (1.4) yields evidently

$$\sum \frac{\partial \Phi}{\partial q_\alpha} \delta q_\alpha \geqslant 0 \tag{4.4}$$

where

$$\Phi (q_1, \ldots, q_m) = \varphi (x_1 (q_1, \ldots, q_m), \ldots, z_n (q_1, \ldots, q_m))$$

Condition (1.6)

$$\Delta T = 0 \tag{4.5}$$

remains unchanged.

The conditions (4.3), (4.4) and (4.5) for the absolutely elastic impact occurring during the application of a single constraint upon a system have been established with the assumption that the considered material system consists of a finite number of material points. These conditions, however, remain in force (axiomatic assumption) for an arbitrary material system with a finite number of degrees of freedom provided its constraints are smooth.

This generalization will be applied below in a case when two solid bodies collide and it will be shown that it (the generalization) is in full agreement with the classical theory of this problem.

**5.** Conditions (4.3), (4.4) and (4.5) possess an interesting geometrical interpretation.

As is known [1], the motion of a holonomic system with constraints independent of time may be geometrically represented as the motion of a point in the $m$-dimensional Riemann configuration space ($m$ is the number of degrees of freedom of the system) the metric of which is determined by the condition

$$\left(\frac{ds}{dt}\right)^2 = 2T$$

where $T$ is the kinetic energy of the system. Let

$$T = \frac{1}{2}\sum a_{\alpha\beta} q_\alpha' q_\beta'$$

Then the metric for the configuration space must be

$$ds^2 = \sum a_{\alpha\beta} dq_\alpha dq_\beta \qquad (5.1)$$

If at some point in the configuration space, two contravariant vectors $l_1, \ldots, l_m$ and $l_1^*, \ldots, l_m^*$ are considered, then their lengths $l$ and $l^*$, as is done in Riemannian geometry, are defined by the equalities

$$l = \sum a_{\alpha\beta} l_\alpha l_\beta, \qquad l^* = \sum a_{\alpha\beta} l_\alpha^* l_\beta^* \qquad (5.2)$$

and the angle $\theta$ between them by the equality

$$ll^* \cos\theta = \sum a_{\alpha\beta} l_\alpha l_\beta^* \qquad (5.3)$$

The vectors are considered orthogonal if $\cos\theta = 0$, i.e.

$$\sum a_{\alpha\beta} l_\alpha l_\beta^* = 0 \qquad (5.4)$$

The velocity of the describing point and the "admissible displacements" of the system are contravariant vectors.

From the definitions (5.2) and (5.3) it can be seen then that the condition (4.5) for the conservation of kinetic energy implies the conservation of the describing point velocity, while the condition

$$\sum \Delta\left(\frac{\partial T}{\partial q_\alpha'}\right)\delta q_\alpha = 0$$

which can also be expressed in the form

$$\sum a_{\alpha\beta} q_\beta' \delta q_\alpha = \sum a_{\alpha\beta} q_{\beta_0}' \delta q_\alpha \qquad (5.5)$$

along with the condition (4.5) implies the equality of angles formed by the direction of an arbitrary "admissible displacement" tangent to the boundary of the region for possible displacement of the system and the velocity of the describing point before impact and immediately after impact.

*Note*. The region $D$ for possible displacements of the system is given by the equation for a single constraint

$$\Phi(q_1 \ldots, q_m) \geqslant 0$$

Consider a unit contravariant vector $l_1, \ldots, l_m$, orthogonal to all "admissible displacements" of the system which are tangent to the boundary of $D$ at the point where the system is bounded by $D$. In accordance with the definition (5.4) its components must satisfy

$$\sum a_{\alpha\beta} l_\beta \delta q_\alpha = 0 \qquad (5.6)$$

for possible $\delta q_1, \ldots, \delta q_m$, satisfying

$$\sum \frac{\partial \Phi}{\partial q_\alpha} \delta q_\alpha = 0 \qquad (5.7)$$

The condition (5.6) yields $l_\beta$, but if it is assumed that $l$ is known then this condition can fully replace condition (5.7). Taking this into account, as well as the fact that the determinant $|a_{\alpha\beta}|$ is different from zero, we find from (5.5) and (5.6)

$$\Delta q_\beta' - \lambda l_\beta = 0 \qquad (5.8)$$

where $\lambda$ is an indetermined multiplier.

Thus, the vector for the increase in the describing-point velocity is colinear with the normal to the boundary of the region $D$ at the point of impact. This implies, on the one hand, that the reflection of the describing point from the boundary occurs in the plane passing through the normal and the describing-point velocity before impact.

On the other hand, substituting Expression (5.8) into condition

$$\Delta \sum a_{\alpha\beta} q_\alpha' q_\beta' = \sum a_{\alpha\beta} q_\beta' \Delta q_\alpha' + \sum a_{\alpha\beta} q_{\alpha_0}' \Delta q_\beta' = 0$$

for the conservation of kinetic energy and reducing the thus-obtained equality by a multiplier different from zero, we obtain

$$\sum a_{\alpha\beta} l_\alpha q_\beta' = - \sum a_{\alpha\beta} l_\alpha q_{\beta_0}'$$

implying the equality of angles for incidence and reflection of the describing point in its reflection from the boundaries of the region $D$.

Thus, if the motion of the material system carries it to the boundary of possible displacements then, as the result of the occurring impact, the system rebounds from the boundary in accordance with the law that "the angle of incidence equals the angle of reflection".

This is especially descriptive in the case when the expression for the kinetic energy of the system is of the form

$$T = \tfrac{1}{2} \sum q_i'^2 \qquad (5.9)$$

since, in this case, the configuration space is Euclidian and the angles assume the usual meaning.

6. Let us apply the theory presented to the solution of some problems.

a) Two material points on a straight line connected by an inextensible string.

When the points, moving away from each other, are at the distance equal to the length of the string, an impact occurs. However, as follows from the theorems on the center of gravity and the moment of momentum, there occur no changes in the motion of the center of gravity or the angular velocity of the string.

b) Two material points of equal mass are placed on a smooth fixed axis.

Let us assume that one of these points is at rest, while the other one is striking it with a certain velocity. The question is, what will be the state of the system immediately after impact if the impact is purely elastic?

The problem permits a purely geometric solution. Indeed, the kinetic energy of the system

$$T = \frac{1}{2} m (x_1'^2 + x_2'^2)$$

where $x_1$ and $x_2$ are the coordinates of the points after the introduction of Lagrangian coordinates $q_1$ and $q_2$ in accordance with

$$q_1 = \sqrt{m} x_1, \qquad q_2 = \sqrt{m} x_2$$

assumes the form (5.9). This means that the space for variables $q_1$ and $q_2$ is Euclidian. It is realized if $q_1$ and $q_2$ are measured along the axes of a Cartesian system of coordinates (say $q_1$ is measured along the abscissa).

As the location of the first point is given, the second point may assume any location on one side of the first point. Assume for definiteness that $x_1 \leqslant x_2$. In accordance with this, the boundary of the region for possible displacements is the straight line $q_1 = q_2$ in the space of the variables $q_1$, $q_2$, while the region itself is the half-plane located to the left and above this straight line.

Let us assume that at the instant of impact the second point in the system is at rest. Then immediately before the impact the velocity of the describing point in the space $q_1$, $q_2$ is directed along the abscissa. The rebounding of the describing point from the boundary is governed by the law that "the angle of incidence equals the angle of reflection". But since the space $q_1$, $q_2$ is Euclidian, the angles take on the usual meaning.

Therefore, immediately after impact the describing point will have the velocity directed along the ordinate.

Thus, if the moving point collides with a stationary point of the same mass, the latter one obtains the velocity of the first one, while the first one stops. This result is well known from the theory of centrally impacting spheres.

c) Consider further the collision of two rigid plane bodies, located and moving in the same plane.

We will assume that the collision of bodies occurs at one point, and that in the neighborhood of the point of contact at least one of the bodies has a smooth contour.

We will choose the system of coordinates such that the origin coincides with the contact point at the moment of impact and that the $x$-axis is directed along the mutual tangent to the contours of the colliding bodies if both contours are smooth, or along the tangent to the smooth contour if there is one such contour. Denote by $x_1$, $y_1$, $\omega_1$ and correspondingly by $x_2$, $y_2$, $\omega_2$ the coordinates of the center of gravity and the angular velocities of the bodies. Let $m_1$ and $m_2$ be the masses, and $c_1$, $c_2$ the moments of inertia of the bodies about their centers of gravity.

Among the "admissible displacements" of the system there are the elementary two-sided translational displacements of each body separately along the $x$-axis and the joint translational displacement of both bodies along the axis. The theorems on the motion of the center of gravity (the particular one along the $x$-axis and the general one along the $y$-axis) give

$$\Delta x_1' = 0, \qquad \Delta x_2' = 0, \qquad \Delta (m_1 y_1' + m_2 y_2') = 0 \tag{6.1}$$

On the other hand, among the "admissible displacements" of the system in these circumstances are the elementary two-sided rotations of each body separately about the points on the $y$-axis, sufficiently distant from the origin of the coordinates. Choose one such point for each body (denoted by $O_1$ and $O_2$). Then, by virtue of the particular theorem on the moment of momentum

$$\Delta K_1 = 0, \qquad \Delta K_2 = 0 \tag{6.2}$$

where $K_1$ and $K_2$ are the moment of momentum of the first body relative to $O_1$ and the moment of momentum of the second body relative to $O_2$, respectively. But

$$K_1 = c_1\omega_1 + m_1 [x_1 y_1' - x_1' (y_1 - a_1)], \qquad K_2 = c_2\omega_2 + m_2 [x_2 y_2' - x_2' (y_2 - a_2)]$$

where $a_1$ and $a_2$ are the ordinates of points $O_1$ and $O_2$, respectively (their abscissas are equal to zero). Substituting these expressions into the equalities (6.2) and taking into account the first two equalities (6.1), we obtain

$$c_1\Delta\omega_1 + m_1x_1\Delta y_1' = 0, \qquad c_2\Delta\omega_2 + m_2x_2\Delta y_2' = 0 \tag{6.3}$$

Equalities (6.3) along with (6.1) and the condition (4.5) give the complete set of equations describing the collision of rigid bodies in the given circumstances.

The study of these equations is beyond the scope of this article. However, we will derive the following proposition, basic to the classical theory of rigid body collisions [3]: the normal component for the relative velocity of the points of contact in impacting rigid bodies changes sign while the magnitude remains unaltered.

We will use the index "zero" for kinematic characteristics of systems immediately before impact, retaining the usual designations for them after impact. Then the condition of conservation of kinetic energy in a system may be written as

$$c_1\Delta\omega_1(\omega_1 + \omega_{10}) + c_2\Delta\omega_2(\omega_2 + \omega_{20}) + m_1\Delta y_1'^2 + m_2\Delta y_2'^2 = 0 \tag{6.4}$$

Utilizing the equalities (6.3), we rewrite (6.4) as

$$-m_1x_1\Delta y_1'(\omega_1 + \omega_{10}) + m_1\Delta y_1'^2 - m_2x_2\Delta y_2'(\omega_2 + \omega_{20}) + m_2\Delta y_2'^2 = 0$$

whence

$$m_1\Delta y_1'[y_1' + y_{10}' - x_1(\omega_1 + \omega_{10})] + m_2\Delta y_2'[y_2' + y_{20}' - x_2(\omega_2 + \omega_{20})] = 0$$

From this and from the equalities (6.1) it follows

$$y_1' + y_{10}' - x_1(\omega_1 + \omega_{10}) - y_2' - y_{10}' + x_2(\omega_2 + \omega_{20}) = 0$$

or

$$(y_1' - x_1\omega_1) - (y_2' - x_2\omega_2) = -[(y_{10}' - x_1\omega_{10}) - (y_{20}' - x_2\omega_{20})]$$

which is as required.

## BIBLIOGRAPHY

1. Synge, J.L., *Tenzornye metody v dinamike (Tensor Methods in Dynamics)*. IL, 1947.

2. Eisenhart, L.P., *Rimanova geometriia* (*Riemannian Geometry*). IIL, 1948.

3. Levi-Civita, T. and Amal'di, U., *Kurs teoreticheskoi mekhaniki* (*Course of Theoretical Mechanics*), Vol. 2, Part 2. IIL, 1951.

*Translated by* V.C.